On Rational Sextic Surfaces Having a Nodal Curve of Order 9.

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1. It is known that, if the section, by a generic plane, of a surface of order greater than 4 is rational, then the surface is ruled.* If the section is of genus 1, Castelnuovo has shown that the surface is either ruled or rational.† Ruled sextics have been the subject of several investigations.‡ In this paper, the non-ruled sextic surfaces whose plane sections are of the lowest possible genus will be studied. These surfaces have been discussed briefly by Caporali.§ Particular cases have been studied by del Pezzo|| and Bonicelli.¶

2. Castelnuovo shows, in the memoir cited above, that the parametric equations

$$x_i = f_i(\lambda_1, \lambda_2, \lambda_3), \qquad i = 1, 2, 3, 4, \tag{1}$$

of the given sextic surface can be set up in such a way that the four curves $f_i = 0$, in the λ-plane, are cubics which have three points P_1, P_2, P_3 in common. These three points are thus fundamental points for the linear system $\Sigma u_i f_i = 0$ of cubics which correspond to the plane sections of the given surface.

If we take the triangle** whose vertices are P_1, P_2, P_3 as coordinate triangle in the λ-plane, equations (1) simplify to

$$x_i = a_{i0}\lambda_1^2\lambda_2 + a_{i1}\lambda_1\lambda_2^2 + b_{i0}\lambda_1^2\lambda_3 + 2b_{i1}\lambda_1\lambda_2\lambda_3 + b_{i2}\lambda_2^2\lambda_3 + c_{i0}\lambda_1\lambda_3^2 + c_{i1}\lambda_2\lambda_3^2, \tag{2}$$
$$i = 1, 2, 3, 4.$$

3. To a fundamental point P_i there corresponds, on the surface, a right line g_i, in such a way that to each direction through P_i corresponds†† a point

* Picard, *Crelle's Journal*, Vol. C, p. 71.

† *Rendiconti dei Lincei*, Series 5, Vol. III (1894), p. 59.

‡ Cf. Snyder, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXVII (1905), pp. 77–102. Other references will be found in this article.

§ "Collectanea Mathematica in Memoriam D. Chelini," p. 169, Hoepli, Pisa, 1881.

|| *Rendiconti di Napoli*, Series 3, Vol III (1897), pp. 196–203.

¶ *Giornali di Mat. di Battaglini*, Series 2, Vol. IX (1897), pp. 184–191.

** For the cases in which P_1, P_2, P_3 do not form a triangle, see Arts. 23–25. Throughout the following discussion, except when the contrary is stated, it should be understood that the general case is the one under discussion.

†† Clebsch, *Math. Ann.*, Vol. I, p. 266.

on g_i. To each direction through $(0, 0, 1)$, for example, corresponds at once, from (2), a point on the line

$$x_i = c_{i0}\lambda_1 + c_{i1}\lambda_2, \qquad i = 1, 2, 3, 4,$$

and conversely. The three lines g_1, g_2, g_3 corresponding to P_1, P_2, P_3 respectively, do not intersect.

To the lines joining P_1, P_2, P_3 in pairs correspond three other lines g'_1, g'_2, g'_3 on the surface. Each of these lines intersects two of the lines g_1, g_2, g_3 so that the six lines form a gauche hexagon.

4. There are no other non-multiple right lines on the surface. For, let g be a line on the surface to which corresponds, in the λ-plane, a curve of order m having an α_i-fold point at P_i $(i=1,2,3)$. Since a generic plane intersects g in a single point, the image cubic of the section of the surface by the plane intersects the image of g in a single non-fundamental point, so that

$$3m - \alpha_1 - \alpha_2 - \alpha_3 = 1.$$

But $\alpha_1 + \alpha_2 + \alpha_3 \leq 2m$, since, otherwise, the curve of order m would be composite. Hence, $m = 1$, $\alpha_1 + \alpha_2 + \alpha_3 = 2$ and g coincides with one of the lines g'_1, g'_2, g'_3.

5. To a generic line in the λ-plane through a fundamental point P_i, corresponds a conic on the surface, since it is intersected by a generic plane in two points. To the pencil of lines through P_i thus corresponds a system of conics on the surface, such that, through a generic point on the surface, there passes a single conic of the system.

The planes of such a system of conics envelope a curve of class 4. For, from (2), as the line $k_2\lambda_1 - k_1\lambda_2 = 0$ describes the pencil with vertex at $\lambda_1 = \lambda_2 = 0$, the plane of the corresponding conic describes the developable of class 4 defined by the equation

$$\begin{vmatrix} x_1 & k_1 a_{10} + k_2 a_{11} & k_1^2 b_{10} + 2k_1 k_2 b_{11} + k_2^2 b_{12} & k_1 c_{10} + k_2 c_{11} \\ x_2 & k_1 a_{20} + k_2 a_{21} & k_1^2 b_{20} + 2k_1 k_2 b_{21} + k_2^2 b_{22} & k_1 c_{20} + k_2 c_{21} \\ x_3 & k_1 a_{30} + k_2 a_{31} & k_1^2 b_{30} + 2k_1 k_2 b_{31} + k_2^2 b_{32} & k_1 c_{30} + k_2 c_{31} \\ x_4 & k_1 a_{40} + k_2 a_{41} & k_1^2 b_{40} + 2k_1 k_2 b_{41} + k_2^2 b_{42} & k_1 c_{40} + k_2 c_{41} \end{vmatrix} = 0. \qquad (3)$$

It will be shown (Art. 20) that, if a sextic surface is generated by a system of non-composite conics whose planes envelope a rational curve of class 4, then a generic plane section of the surface is of genus 1.

Four tangents can be drawn from each of the points P_i to a generic curve $\Sigma u_i f_i = 0$. Hence, four conics of each of the three systems touch a generic plane. The cross-ratios of the parameters of the four conics of the three systems are respectively equal.

6. We have seen that three conics, one of each of the above systems, pass through a generic point on the surface. Through such a point, no other conic lying on the surface can pass. For, let C_2 be a non-multiple conic on the surface and let its image on the λ-plane be of order m and have an α_i-fold point at the fundamental point $P_i\,(i=1,2,3)$. We have (cf. Art. 4)

$$3m-\alpha_1-\alpha_2-\alpha_3=2.$$

Since $\alpha_1+\alpha_2+\alpha_3\leq 2m$, we find either $m=1$, $\alpha_1+\alpha_2+\alpha_3=1$, or $m=2$, $\alpha_1+\alpha_2+\alpha_3=4$. But the latter case is impossible, since a proper conic can not have a node. Hence, C_2 belongs to one of the three given systems.

7. Koenigs has shown* that if, through a generic point on a surface, there pass two (or more) conics lying on the surface, then the surface is rational and a generic plane section is of genus not greater than unity. If such a surface is a sextic, it can not be ruled, since it contains infinitely many non-multiple conics. Hence, the plane sections are of genus 1 (Art. 1) and the surface is generated by precisely three systems of conics.

8. There are two systems of cubics, each ∞^2, on the given surface. Those of the first system correspond to the right lines in the λ-plane and intersect the lines g_1', g_2', g_3'; those of the second system correspond to the conics through P_1, P_2, P_3 and intersect g_1, g_2, g_3. The two systems are equivalent, since, if we subject the points of the λ-plane to the transformation

$$\lambda_i=\frac{1}{\lambda_i'}, \qquad i=1,2,3,$$

the representations of the two systems on the surface (2) are interchanged.

There are three systems, each ∞^3, of quartics. They correspond to the systems of conics through two fundamental points, and are thus characterized by the non-intersection of one line of either triad g_1, g_2, g_3 or g_1', g_2', g_3'.

There are six systems, each ∞^4, of quintics. They correspond to the conics through a fundamental point and to the cubics with a double point at one fundamental point and passing through the other two fundamental points.

There are eight systems, each ∞^5, of rational sextics. They correspond to the conics in the λ-plane, to the quartics with double points at P_1, P_2, P_3, and to the cubics which have a node at one of these points and pass through a

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* *Annales de L'École Normale Supérieure,* Series 3, Vol. V. Koenigs' proof expressly excludes the case of a non-rational surface generated by a system of conics such that three (or more) conics of the system pass through a generic point of the surface and two generic conics of the system have just one variable point in common. No such surface exists, however, since it follows from a theorem by Castelnuovo (*Atti di Torino,* Vol. XXVIII (1893), p. 736) that a surface generated by such a system of conics is rational.

second. There is also a system, ∞^6, of sextics of genus 1 corresponding to the cubics through P_1, P_2, P_3 (cf. Art. 13).

9. *The genus a non-multiple curve of order n on the given surface does not exceed the greatest integer in*

$$\frac{n^2-6n+12}{12}.$$

Let C_n be the given curve and let the corresponding curve C_m in the λ-plane be of order m and have an α_i-fold point at $P_i (i=1,2,3)$. We have (cf. Art. 4)

$$3m-\alpha_1-\alpha_2-\alpha_3=n.$$

Let $\alpha_1+\alpha_2+\alpha_3=k$, so that $m=\dfrac{n+k}{3}$. The genus of C_n, which is equal to that of C_m, does not exceed

$$\frac{1}{18}(n+k-3)(n+k-6)-\frac{1}{6}k(k-3).$$

The maximum value of this expression is $\dfrac{n^2-6n+12}{12}$.

The limit stated in the above theorem is actually attained for all values of n.

10. The characteristic numbers for the given surface can be determined at once from formulas by Caporali.*

The surface is of class 12. The tangent cone from a generic point to the surface is of order 12 and genus 7. The parabolic curve is of order 24 and genus 25. The tangent planes along the parabolic curve envelop a curve of class 24 and order 48.

If we form the invariants S and T for the cubics $\Sigma u_i f_i=0$ which correspond to the plane sections of the given surface, we determine two surfaces, $S(u_1, u_2, u_3, u_4)=0$ and $T(u_1, u_2, u_3, u_4)=0$, of class 4 and 6 respectively, enveloped by the planes which intersect the surface in curves for which the corresponding invariants are zero. The planes tangent to both of these surfaces touch the given surface along the parabolic curve.

The envelope of the planes which intersect the given surface in curves for which the corresponding absolute invariant is equal to a given constant k is a surface $S^3-kT^2=0$ of class 12. In particular, if $k=6$, so that the section is rational, we have the equation

$$S^3-6T^2=0$$

of the given surface in plane coordinates.

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* "Collectanea Mathematica in Memoriam D. Chelini," 1881. A few of the results for this surface are explicitly stated (p. 169).

11. The double developable of the given surface is of class 24. The section of the surface by a bitangent plane is composite, since it has eleven double points. The developable of these planes consists of the six pencils having the six lines on the surface (Art. 3) as axes, the three developables of class 4 determined by the planes of the three systems of conics on the surface, and a developable of class 6 and genus 1 whose planes intersect the surface in two cubics. To two coplanar cubics correspond a line and a conic in the λ-plane; so two such cubics belong to distinct systems of cubics on the surface (Art. 8). The cubics in the planes of the sextic developable thus constitute two systems (each of genus 1) of rational plane cubics on the surface. Three cubics of each system pass through a generic point on the surface; each cubic has a double point on the double curve and intersects the double curve in seven other points. At each point of the double curve, one cubic of each system has a double point.

12. The double curve C_9 is of order 9 and genus 1. It has four triple points which are also triple on the surface. There are twelve pinch-points on C_9. The corresponding curve in the λ-plane is of order 9. It has a triple point at each fundamental point and no other point singularities.

The double curve C_9 lies on a cubic surface. For, since C_9 is of genus 1, the pencil of cubic surfaces through the triple points and fourteen generic points of C_9 all intersect C_9 in a fixed point. The cubic of the pencil which passes through another generic point of C_9 contains the curve. The double curve constitutes the complete intersection of this cubic surface with the given surface.

13. Every curve of order n on the given surface intersects C_9 in $3n$ points, since its $3n$ intersections with a cubic surface lie on C_9. In particular, the sextics of genus 1 which correspond to the cubics through P_1, P_2, P_3 (Art. 8) intersect C_9 in eighteen points. Hence, each of these curves lies on a quartic surface which contains C_9, since the quartic surface which contains the triple points and twenty-four generic simple points of C_9 and six generic points of such a sextic contains both curves. Each cubic through P_1, P_2, P_3 thus defines a quartic surface which contains C_9, and conversely.

The quartic surfaces which contain C_9 and a fixed cubic C_3 on the surface constitute a bundle of quartics which cut from the surface the system of cubics opposite to the one to which C_3 belongs (Art. 8). Since two cubics of the same system intersect in a single point, the residual intersection of two quartics of the bundle determines a unique point on the surface, and conversely, so that the parametric equations of the surface can be determined as soon as such a bundle of quartics has been found.

14. The cubic curves in the λ-plane which pass through P_1, P_2, P_3 form a linear system ∞^6 and thus determine a surface of order 6 belonging to* an S_6,

$$\xi_i = f_i(\lambda_1, \lambda_2, \lambda_3), \qquad i = 1, 2, 3, \ldots, 7, \tag{4}$$

where $\xi_1, \xi_2, \ldots, \xi_7)$ are current coordinates and the curves $f_i(\lambda) = 0$ are cubics through P_1, P_2, P_3. Del Pezzo † has shown that the equations of any non-ruled sextic surface belonging to an S_6 can be written in the above form.

By a suitable choice of coordinate system, the fundamental points P_1, P_2, P_3 being supposed to form a triangle (Art. 2), equations (4) can be reduced to

$$\xi_1=\lambda_1^2\lambda_2,\ \xi_2=\lambda_1\lambda_2^2,\ \xi_3=\lambda_2^2\lambda_3,\ \xi_4=\lambda_2\lambda_3^2,\ \xi_5=\lambda_3^2\lambda_1,\ \xi_6=\lambda_3\lambda_1^2,\ \xi_7=\lambda_1\lambda_2\lambda_3. \tag{5}$$

The surface of equations (2) is the projection on S_3, from a generic plane π, of the surface (5).

15. Bordiga has shown ‡ that the point of intersection of corresponding S_4 of three projective bundles of S_4 belonging to an S_6 is a non-ruled sextic surface. Such a surface can, in fact, be generated in infinitely many ways by such a correspondence. For, as in Art. 13, it is seen that the S_4 of the bundle in S_6 determined by an S_3 which contains a cubic curve on the surface are in (1, 1) correspondence with the points of the surface, and hence with the points of the λ-plane. Three such bundles are thus in projective correspondence with each other, in such a way that corresponding S_4 intersect on the surface.

In particular, we can take the basis S_3 of each of the three bundles to contain two non-intersecting lines on the surface. The surface can thus be defined by the three projective bundles

$$\begin{matrix} \lambda_3\xi_1 = \lambda_1\xi_7 \\ \lambda_3\xi_2 = \lambda_2\xi_7 \end{matrix}, \quad \begin{matrix} \lambda_1\xi_3 = \lambda_2\xi_7 \\ \lambda_1\xi_4 = \lambda_3\xi_7 \end{matrix}, \quad \begin{matrix} \lambda_2\xi_5 = \lambda_3\xi_7 \\ \lambda_2\xi_6 = \lambda_1\xi_7 \end{matrix}.$$

16. The ∞^4 bisecants of (5) generate an hypersurface H in S_6. This hypersurface is of order 3 and has the surface (5) as a double surface. For, the plane π' determined by three generic points of the surface (5) contains the three lines, lying on H, which join these three points in pairs. It has no other points in common with H, since the projection of (5) from π' on S_3 is a cubic surface with no multiple points.

The hypersurface H is generated by each of two rational systems of ∞^2 S_3 such that each S_3 contains a cubic curve lying on the surface (5). For, let C_3 be a cubic curve of either of the two systems (cf. Art. 8) lying on (5). The S_3 determined by C_3 lies on H since, through a given point of S_3, a bisecant

* The notation S_r is used to indicate a space of r dimensions. An entity is said to belong to an S_r if it lies entirely in the S_r but not in any S_{r-1}.

† *Rendiconti di Palermo*, Vol. I, p. 441.

‡ *Comptes Rendus*, Vol. CII, pp. 743–745.

of C_3 can be passed. Each generator of H lies in an S_3 of the system determined by C_3, since, through its two points of intersection with (5), a cubic of the given system can be passed.

Two generic S_3 on H, of opposite systems, have a line in common, since their corresponding cubics intersect in two points. Two S_3 of the same system have in common a single point, the point of intersection of their corresponding cubics.

17. The surface (5) has no trisecant lines, since its projection on S_4 from such a line would be a cubic surface belonging to S_4 and having its generic hyperplane sections of genus 1, which is impossible.

18. The points of the double curve of (2) are in (1,1) correspondence with the generators of H which intersect the plane π (Art. 14) from which (5) is projected onto (2). These generators define a ruled surface of order 12 and genus 1 belonging to S_6. Twelve of its generators touch (5), since there are twelve pinch-points of (2) on the double curve (Art. 12). It follows that the four-spread generated by the tangent planes to (5) is of order 12. It has the surface (5) as fourfold surface.

19. It follows* from Art. 5 that the surface (2) can be defined (in any one of three ways) as the locus of the conic of intersection of corresponding surfaces of the two systems

$$L_1 k_1^4 + L_2 k_1^3 k_2 + L_3 k_1^2 k_2^2 + L_4 k_1 k_2^3 + L_5 k_2^4 = 0, \tag{6}$$

$$Q_1 k_1^3 + Q_2 k_1^2 k_2 + Q_3 k_1 k_2^2 + Q_4 k_2^3 = 0, \tag{7}$$

where k_1 and k_2 are parameters, $L_i = 0$ $(i=1, 2, \ldots, 5)$ is a plane, and $Q_i = 0$ $(i=1, 2, 3, 4)$ is a quadric surface.

Since the given surface is of order 6, there are five values of the ratio $k_2 : k_1$ for which the plane (6) is a component of the corresponding quadric (7). It is no restriction to suppose that these five values coincide at $k_2 = 0$. For, if

$$Q_1 \alpha^3 + Q_2 \alpha^2 + Q_3 \alpha + Q_4 \equiv L(L_1 \alpha^4 + L_2 \alpha^3 + L_3 \alpha^2 + L_4 \alpha + L_5),$$

where $L=0$ is a plane, then $k_1 - \alpha k_2$ is a factor of the left member of

$$L(L_1 k_1^4 + L_2 k_1^3 k_2 + L_3 k_1^2 k_2^2 + L_4 k_1 k_2^3 + L_5 k_2^4) \\ - k_2 (Q_1 k_1^3 + Q_2 k_1^2 k_2 + Q_3 k_1 k_2^2 + Q_4 k_2^3) = 0.$$

If we remove the factor $k_1 - \alpha k_2$, the resulting equation, with (6), determines the given system of conics. If we replace (7) by the equation so determined, and repeat the above operation successively, we determine, in place of (7), a system of quadrics

$$L_1 L_5' k_1^3 + (L_2 L_5' + L_1 L_4') k_1^2 k_2 + (L_3 L_5' + L_2 L_4' + L_1 L_3') k_1 k_2^2 \\ + (L_4 L_5' + L_3 L_4' + L_2 L_3' + L_1 L_2') k_2^3 = 0, \tag{8}$$

* Cf. also an article by the author in AMERICAN JOURNAL OF MATHEMATICS, Vol. XXX, pp. 99–116.

where $L'_i=0\,(i=1,2,\ldots,5)$ is a plane and the ten planes $L_i=0$, $L'_i=0$ satisfy the identity

$$L_1L'_1+L_2L'_2+L_3L'_3+L_4L'_4+L_5L'_5\equiv 0. \tag{9}$$

From (6), (8) and (9) it follows that the given system of conics is also defined by (6) and any one of the systems of quadrics

$$L_1L'_4k_1^3+(L_2L'_4+L_1L'_3)k_1^2k_2+(L_3L'_4+L_2L'_3+L_1L'_2)k_1k_2^2-L_5L'_5k_2^3=0, \tag{10}$$

$$L_1L'_3k_1^3+(L_2L'_3+L_1L'_2)k_1^2k_2-(L_4L'_4+L_5L'_5)k_1k_2^2-L_5L'_4k_2^3=0, \tag{11}$$

$$L_1L'_2k_1^3+(L_2L'_2+L_1L'_1)k_1^2k_2-(L_4L'_5+L_3L'_4)k_1k_2^2-L_5L'_3k_2^3=0. \tag{12}$$

If we eliminate k_1 and k_2 between (8), (10), (11) and (12), remove extraneous linear factors, and simplify, we find, as the equation of the surface,

$$\begin{vmatrix} L_1L'_3 & L'_4 & L'_5 & Q_2 \\ L_1L'_2 & L'_3 & L'_4 & L_5L'_5 \\ L_1L'_1 & L'_2 & L'_3 & L_5L'_4 \\ Q_1 & L'_1 & L'_2 & L_5L'_3 \end{vmatrix}=0, \tag{13}$$

wherein has been written, for brevity,

$$Q_1\equiv-(L_5L'_4+L_4L'_3+L_3L'_2+L_2L'_1),\quad Q_2\equiv-(L_4L'_5+L_3L'_4+L_2L'_3+L_1L'_2).$$

20. At points on the double curve, equations (8), (10), (11) and (12) have two solutions in common. Hence, all the first minors of (13) are zero for points on this curve. The linear system ∞^6 of quartic surfaces which contain the double curve (Art. 13) are determined by the quartic surfaces in the above set.

At the triple points of the surface, all the second minors of (13) are zero.

The equation of the cubic surface (Art. 12) on which the double curve lies is

$$\begin{vmatrix} L'_1 & L'_2 & L'_3 \\ L'_2 & L'_3 & L'_4 \\ L'_3 & L'_4 & L'_5 \end{vmatrix}=0. \tag{14}$$

It follows, in fact, from (9) that the complete intersection of (13) and (14) is a double curve on (13).

21. Denote the left member of (13) by f_6 and of (14) by f_3. The double curve of $f_6=0$ is also double on all the sextic surfaces of the pencil

$$k_1f_6+k_2f_3^2=0. \tag{15}$$

Every sextic surface for which this curve is double belongs to the pencil (15), since the double curve forms its complete intersection with the surfaces (15).

Let there be given a space curve C_9 of order 9 and genus 1 which has four triple points (Art. 12). There exists a pencil of sextic surfaces on which

C_9 is a double curve. For, if we take the triple points of C_9 as vertices of the coordinate tetrahedron, and subject C_9 to the cubic transformation

$$x_i = \frac{1}{x_i'}, \qquad i=1, 2, 3, 4, \tag{16}$$

then C_9 is transformed into a plane cubic C_3. The sextic surfaces on which C_3 is a double curve are of the form

$$\phi_1^2\phi_4+\phi_1\phi_2\phi_3+\phi_3^2=0,$$

where $\phi_1=0$ is the plane of C_3, $\phi_3=0$ is the cubic cone projecting C_3 from a fixed point P, $\phi_2=0$ is a quadric cone with vertex at P, and $\phi_4=0$ is a quartic surface. If we subject the forty-two independent homogeneous parameters in this equation to the forty conditions that the surface shall have a triple point at each vertex of the tetrahedron of reference, we determine a pencil of sextics which are transformed by the involutorial transformation (16) into a pencil of sextics which have C_9 as double curve.

It follows from a theorem by Halphen* that twelve sextics of the pencil (15) touch a generic plane.

22. The condition that a curve of order n, not lying on $f_3=0$, shall lie on a surface of the pencil (15) is that its $3n$ intersections with $f_3=0$ shall lie on the double curve C_9 (Art. 13). The locus of the right lines which lie on surfaces (15) is thus the scroll of trisecants R of C_9. The surface R has C_9 as eightfold curve, since the projecting cone of C_9 from a generic point on the curve has eight double generators. The order of R is 25, since its complete intersection with a generic surface (15) consists of C_9 and six lines. The triple points of C_9 are twelvefold points of R. In fact, the plane determined by each pair of tangents to C_9 at a triple point P_3 is torsal tangent plane to R along the generators joining P_3 to each of the four distinct intersections of C_9 with the plane.

Through a generic point in space, there pass three conics which intersect C_9 in six points (Art. 5). These three conics lie on a quartic surface which contains C_9. The ∞^1 cubics through a generic point P which have nine points in common with C_9 constitute two systems (Art. 8) each of which generates the surface (15) through P. Cubics through P, of opposite systems, intersect in one variable point.

23. Some particular cases of the surface defined by equations (1) will now be noticed.

We have supposed (Art. 2) that the fundamental points P_1, P_2, P_3 in the λ-plane form a triangle. If two of these points, P_1, P_2, are consecutive, the

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* *Bull. Soc. Math.*, Vol. X (1882), p. 166.

surface has an additional double point D determined by the bundle of planes whose corresponding cubics have a double point at P_1. Two lines of each triad on the surface (Art. 3) are consecutive and pass through D. Two of the three generating systems of conics coincide. All the conics of the coincident systems pass through D. Conversely, if all the conics of a system on the surface pass through a point, this point is a double point additional to the double curve and the given system counts for two. For, let P_i be the vertex of the corresponding pencil. The point common to the conics corresponds either to P_i on all lines through P_i or else to a curve intersecting these lines in at least one point distinct from P_i. In the latter case we have, using the notation of Art. 4,

$$3\,m-\alpha_1-\alpha_2-\alpha_3=0, \qquad \alpha_i<m,$$

which is impossible. Hence, to all directions through P_i, corresponds a fixed point on the surface so that (Cf. Art. 3) a second fundamental point coincides with P_i.

24. If the fundamental points P_1, P_2, P_3 lie on a line l, then all the points of l correspond to a fixed double point D on the surface, since every cubic of the system $\Sigma\, u_i f_i=0$ which contains a point of l distinct from P_1, P_2, P_3 has l as a component. The six lines on the surface pass through D and coincide in pairs. The systems of generating conics are distinct. The two composite conics of each system are consecutive and pass through D. The two systems of cubic curves on the surface (Art. 8) coincide. All these cubics pass through D.

25. The surfaces of Arts. 23 and 24 are projections of sextic surfaces belonging to S_6 (Art. 14) each of which has a conical double point. A non-ruled sextic surface belonging to S_6 (and hence its sextic projection on S_3) can have at most two (distinct) conical double points. This happens when P_1, P_2, P_3 are collinear and two of them coincide. One of these double points is of the type of Art. 23, the other of Art. 24.

26. If the given surface (2), in S_3, has a double right line, l_2, then, since the surface is not ruled, the residual sections in the planes through l_2 are either (a) rational quartics or (b) pairs of conics. In case (a), to the pencil of quartics on the surface corresponds, in the λ-plane, a pencil of conics through two fundamental points, P_2, P_3. To l_2 thus corresponds a right line through P_1. It is thus a degenerate conic of the system determined by the pencil of lines through P_1. It follows from (3) that the planes of these conics envelope a curve of class 3, and that, conversely, if the planes of such a system envelope a curve of class 3, a conic of the system degenerates into a double line. The line l_2 passes through two triple points on the surface corresponding to

the non-fundamental basis-points of the pencil of conics determined by the residual sections in the planes through l_2. It intersects the residual nodal curve in one other point.

In case (b), the conics in the planes through l_2 all belong to one system, since through each point of the surface there passes a conic of the system. Each plane through l_2 thus determines two lines through a fundamental point P_1, and l_2 is determined by the coincidence of the line corresponding to P_1 with the line corresponding to the join of P_2 and P_3. Since the developable (3) of the conics in the planes through l_2 reduces to a pencil of planes counted twice, there are two double lines of type (a) which intersect l_2 on this surface.

The surface (a) is the projection of the surface (5) from a plane which intersects in a point the plane of a conic on (5). The surface (b) is the projection of (5) from a plane which intersects in a line the S_3 determined by a pair of opposite sides of the hexagon formed by the lines on* (5).

27. The condition that a non-ruled sextic, whose plane sections are of genus 1, have a fourfold line, is that it have three double lines of type (a) (Art. 26) all of which belong to the same system of conics on the surface. To the fourfold line corresponds a conic through two fundamental points. The three double lines, and four of the simple lines, intersect the fourfold line. Each of the other simple lines intersects the three double lines. The double lines may be consecutive to each other or to the fourfold line.

28. If the multiple curve reduces to a triple cubic, this cubic can not have any apparent double points, since the bisecant of the triple curve from a generic point on the surface would lie on it so that the surface would be ruled. The triple curve thus reduces to three lines concurring in a fivefold point of the surface. The tangent cone at the fivefold point intersects the surface in the triple curve and in three lines each of which counts for two consecutive simple lines, so that the surface is of the type of Art. 24. The variable intersection with the surface of the pencil of quadric cones determined by the three triple lines and any one of the simple lines, is a system of conics on the surface.

29. If the given surface has a triple curve of order 2, that curve can not be two skew lines, since the triple curve has no apparent double points (Art. 28). The surface is transformed, by a quadratic transformation which has the triple curve and a simple point on the surface as fundamental elements, into a quintic

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* Del Pezzo has discussed [*Rendiconti di Napoli,* Series 3, Vol. III (1897), pp. 196–203)] a sextic surface with nine double lines. Six of these are of type (a) and concur in a fourfold point. The other three are coplanar and of type (b). The equation of the surface is

$$(x_1+x_2+x_3)\,(x_1+x_2-x_3)\,(x_1-x_2+x_3)\,(x_1-x_2-x_3)\,x_4^2=x_1^2\,x_2^2\,x_3^2\,.$$

surface which has the inverse fundamental conic as double conic and an additional double cubic.

30. If the inverse double conic is not composite, it is known* that the residual double cubic on the quintic degenerates into three lines which intersect the double conic and concur at a threefold point on the surface. Hence, if the given sextic has a proper triple conic, the residual nodal curve is three double lines which concur at a triple point.† The pencil of quadrics determined by the triple conic and any two double lines determines a system of conics on the surface to which the third double line belongs. These lines are thus of type (a) (Art. 26).

31. If the inverse double conic is composite, the point of intersection of its components is a triple point, at least, on the quintic. The residual‡ nodal curve is either a proper cubic through the triple point, a right line through the triple point and a conic, or three lines through the intersection of the components of the fundamental conic. This point is, in this case, a fourfold point on the surface.

It follows at once that if the triple curve on the sextic is two (distinct or consecutive) intersecting lines, the point of intersection of these lines is a fourfold point (at least) on the surface. The residual multiple curve is either a proper cubic through the fourfold point, a line through the fourfold point and a conic, or three lines through a fivefold point.

The multiple curve, in Arts. 30 and 31, is the complete intersection of the given surface with a quadric.

32. If the given surface has a single triple line, the parametric equations of the surface can be chosen so that the curve in the λ-plane corresponding to the triple line is a conic through the fundamental points. To the residual cubics of section by the planes through the triple line correspond the lines of a pencil. To the vertex of this pencil corresponds a fourfold point of the surface which lies on the triple line. The double curve of order 6 is rational. It has, at the fourfold point, a triple point such that the three tangents are coplanar. The cubic surface (Art. 20) on which the multiple curve lies is a cone with vertex at the fourfold point. It has the triple line as double generator.

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* Cf. an article by the author in AMERICAN JOURNAL OF MATHEMATICS, Vol. XXX, p. 110.

† This surface was studied by Bonicelli, *Giornali di Mat. di Battaglini*, Series 2, Vol. IX (1902), pp. 184–191.

‡ Cf. first footnote, Art. 30. The quintic with five double lines was erroneously omitted from the classification there referred to. The equation of such a surface is

$$(x_1^2+x_2^2+x_3^2)^2 x_4 = \phi_5(x_1, x_2, x_3),$$

where $\phi_5 = 0$ is the equation of a quintic cone with five double generators which lie on $x_1^2+x_2^2+x_3^2=0$.